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Matroidal structure of rough sets and its characterization to attribute reduction

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ABSTRACT

Rough sets are efficient for data pre-processing in data mining. However, some important problems such as attribute reduction in rough sets are NP-hard, and the algorithms to solve them are almost greedy ones. As a generalization of the linear independence in vector spaces, matroids provide well-established platforms for greedy algorithms. In this paper, we apply matroids to rough sets through an isomorphism from equivalence relations to 2-circuit matroids. First, a matroid is induced by an equivalence relation. Several equivalent characterizations of the independent sets of the induced matroid are obtained through rough sets. Second, an equivalence relation is induced by a matroid. The relationship between the above two inductions is studied. Third, an isomorphism from equivalence relations to 2-circuit matroids is established, which lays a sound foundation for studying rough sets using matroidal approaches. Finally, attribute reduction is equivalently formulated with rank functions and closure operators of matroids. These results show the potential for designing attribute reduction algorithms using matroidal approaches.

1. Introduction

With the advent of huge data, knowledge analysis and disposal technology become increasingly important. As one of those important techniques, rough set theory proposed by Pawlak [24] attracts much research interest. In theory, axiomatic systems have been constructed [1,28,33,40,44], generalization works have been done [26,27,37,41,45] and connections with other theories and methods have been built [5,6,39,43]. In application, rough set theory has been widely used in attribute reduction [2,20,25,38] and rule extraction [4,23,29].

However, many important problems including attribute reduction in rough sets are NP-hard. Hence those algorithms for them are almost greedy ones [7–9,20], especially heuristic ones [10,19,21,30]. In order to establish better mathematical structures and seek efficient approaches for those problems, rough set theory has been combined with other theories, such as topology [12,15,42], lattices [18,31], fuzzy sets [11,22,34] and matroids [16,32,46]. Specifically, the matroid [14,17] borrows extensively from linear algebra and graph theory, so it is an important mathematical structure with high applicability. Matroids have been applied to diverse fields such as algorithm design, combinatorial optimization and integer programming. Especially, they provide well-established platforms for greedy algorithms. Therefore, the establishment of matroidal structures of rough sets may be much helpful for those NP-hard problems.

In this paper, a matroidal structure of rough sets is constructed, then attribute reduction of information systems is equivalently represented by matroidal approaches. First, a matroid is induced by an equivalence relation through the circuit axioms, and several equivalent formulations of the independent sets of the matroid are provided using rough sets. Second, an equivalence relation is induced by a matroid. The relationship between those two inductions is studied, and a type of matroid, namely 2-circuit matroid, is defined. An isomorphism from equivalence relations to 2-circuit matroids is constructed, which provides a basis for investigating rough sets through matroids. Third, the lower and upper approximations in rough sets are equivalently represented with matroidal approaches. Fourth, attribute reduction of information systems is also characterized by matroids induced by equivalence relations generated by several attributes.

The rest of this paper is organized as follows. Section 2 reviews some fundamental concepts about rough sets, information systems and matroids. In Section 3, a matroid is induced by an equivalence relation. Section 4 studies rough sets by matroids induced by equivalence relations. In Section 5, information systems are investigated in matroidal structures. Finally, Section 6 concludes this paper and points out further works.

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2. Basic definitions

This section recalls some fundamental definitions related to rough sets, information systems and matroids.

2.1. Rough set model

Rough set theory provides a systematic approach to data preprocessing in data mining. In information/decision systems, any attribute subset is characterized by an equivalence relation. A universe together with an equivalence relation on the universe forms an approximation space.

Definition 1 (Approximation space [36]). Let U be a nonempty and finite set called universe. Let R be an equivalence relation on U, i.e., R is reflexive, symmetric and transitive. The ordered pair (U,R) is called an approximation space.

In rough sets, a pair of approximation operators are used to describe an object. In the following definition, a widely used pair of approximation operators are introduced.

Definition 2 (*Approximation* operator [36]). Let R be an equivalence relation on U. A pair of approximation operators $\underline{R},\overline{R}: 2^U \to 2^U$, are defined as follows: for all $X \subseteq U$,

$$\underline{R}(X) = \{x \in U | RN(x) \subseteq X\},$$
$$\overline{R}(X) = \{x \in U | RN(x) \bigcap X \neq \emptyset\},$$

where $RN(x) = \{y \in U | xRy\}$. They are called the lower and upper approximation operators with respect to R, respectively.

In an approximation space, a set is called a precise set if it can be precisely described by an equivalence relation; otherwise, it is called a rough set.

Definition 3 (*R-precise* and R-rough set [36]). Let R be an equivalence relation on U. For all $X \subseteq U$, if $\overline{R}(X) = \underline{R}(X)$, then we say X is a R-precise set; otherwise, we say X is a R-rough set.

2.2. Information system

In many real world applications, information and knowledge are stored, represented and observed in information systems, where a set of objects is characterized by a set of attributes. From a theoretical point of view, an information system refers to two-tuples consisting of the considered objects called universe, and the attributes used to represent those objects.

Definition 4 (*Information* system [13]). An information system is an ordered pair $IS = (U,A)$, where U is a nonempty finite set of objects and A is a nonempty finite set of attributes such that $a : U \to V_a$ for all $a \in A$, where V_a is called the value set of a.

The following definition shows that any attribute subset is represented by an equivalence relation, which lays a sound foundation for investigating information systems using rough set models.

Definition 5 (*Indiscernibility* relation [13]). Let $IS = (U,A)$ be an information system. For all $B \subseteq A$,

$$IND(B) = \{(x,y) \in U \times U | \forall b \in B, b(x) = b(y)\}$$

is called the indiscernibility relation induced by B, simply denoted by B.

In fact, the relation induced by any attribute subset is an equivalence relation. Therefore, the universe together with an equivalence relation induced by some attributes forms an approximation space. Therefore, rough set model is available to study information systems.

Proposition 1 [13]. *Let $IS = (U,A)$ be an information system. For all $B \subseteq A, IND(B)$ is an equivalence relation on U.*

Large scale data contain many redundant data. And there is much need to remove those redundant data in order to obtain useful data in an efficient manner. For this purpose, reducts of an information system are defined.

Definition 6 (*Reduct* [13]). Let $IS = (U,A)$ be an information system. For all $B \subseteq A, B$ is called a reduct of IS, if it satisfies the following two conditions:

(1) For all $b \in B, IND(B) \neq IND(B - \{b\})$;
(2) $IND(B) = IND(A)$.

2.3. Matroids

Matroids are algebraic structures that generalize linear independence in vector spaces. They have a variety of applications in integer programming, combinatorial optimization, algorithm design, and so on. In the following definition, one of the most valuable definitions of matroids is presented in terms of independent sets.

Definition 7 (*Matroid* [14])). A matroid is an ordered pair $M = (U,\mathbf{I})$ where U (the ground set) is a finite set, and $\mathbf{I}$ (the independent sets) a family of subsets of U with the following properties:

(I1) $\emptyset \in \mathbf{I}$;
(I2) If $I \in \mathbf{I}$, and $I' \subseteq I$, then $I' \in \mathbf{I}$;
(I3) If $I_1, I_2 \in \mathbf{I}$, and $|I_1| < |I_2|$, then there exists $u \in I_2 - I_1$ such that $I_1 \bigcup \{u\} \in \mathbf{I}$, where $|I|$ denotes the cardinality of I.

Matroids developed mainly from a comprehensive research on the properties of linear independence and dimension in vector spaces. In order to illustrate the fact that linear algebra is an original source of matroid theory, an example is provided from the viewpoint of the linear independence in vector spaces.

Example 1. Let $U = \{a_1,a_2,a_3,a_4\}$ where $a_1 = [100]^T, a_2 = [010]^T, a_3 = [001]^T$ and $a_4 = [-10-1]^T$. Denote $\mathbf{I} = \{X \subseteq U | X$ are linearly independent$\}$, i.e., $\mathbf{I} = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1,a_2\}, \{a_1,a_3\}, \{a_1,a_4\}, \{a_2,a_3\}, \{a_2,a_4\}, \{a_3,a_4\}, \{a_1,a_2,a_3\}, \{a_1,a_2,a_4\}, \{a_2,a_3,a_4\}\}$. Then $M = (U,\mathbf{I})$ is a matroid.

The above example shows that the independent set of a matroid is a generalization of the linearly independent set; in other words, matroid theory and linear algebra coincide with each other when the independence is degenerated to the linear independence.

In order to indicate that graph theory is another original source of matroids, an example is presented from the cycle of a graph.

Example 2. Let $G = (V,E)$ be the graph as shown in Fig. 1. Denote $\mathbf{I} = \{I \subseteq E | I$ does not contain a cycle of $G\}$, i.e., $\mathbf{I} = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1,a_2\}, \{a_1,a_3\}, \{a_1,a_4\}, \{a_2,a_3\}, \{a_2,a_4\}, \{a_3,a_4\}, \{a_1,a_2,a_4\}, \{a_1,a_3,a_4\}, \{a_2,a_3,a_4\}\}$. Then $M = (E,\mathbf{I})$ is a matroid, where $E = \{a_1,a_2,a_3,a_4\}$.

If a subset of the ground set is not an independent set of a matroid, then it is called a dependent set of the matroid. Based on the dependent set, we introduce the circuit of a matroid. For this purpose, several denotations are presented.

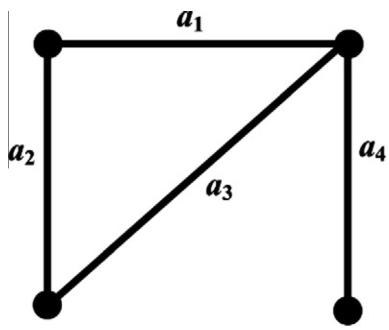

Fig. 1. A graph.

Definition 8 [14]. Let $\mathbf{A}$ be a family of subsets of U. One can denote

$Upp(\mathbf{A}) = \{X \subseteq U | \exists A \in \mathbf{A}, such that\ A \subseteq X\}$,

$Low(\mathbf{A}) = \{X \subseteq U | \exists A \in \mathbf{A}, such that\ X \subseteq A\}$,

$Max(\mathbf{A}) = \{X \in \mathbf{A} | \forall Y \in \mathbf{A}, X \subseteq Y \Rightarrow X = Y\}$,

$Min(\mathbf{A}) = \{X \in \mathbf{A} | \forall Y \in \mathbf{A}, Y \subseteq X \Rightarrow X = Y\}$.

The dependent set of a matroid generalizes the linear dependence in vector spaces and the cycle in graphs. The circuit of a matroid is a minimal dependent set.

Definition 9 (*Circuit* [14]). Let $M = (U, \mathbf{I})$ be a matroid. A minimal dependent set in M is called a circuit of M, and we denote the family of all circuits of M by $\mathbf{C}(M)$, i.e., $\mathbf{C}(M) = Min(\mathbf{I}^c)$, where $\mathbf{I}^c$ is the complement of $\mathbf{I}$ in 2^U.

An example is provided to illustrate that the circuit of a matroid generalizes the cycle of a graph.

Example 3 (Continued from Example 2). The family of circuits of $M = (E, \mathbf{I})$ is $\mathbf{C}(M) = \{\{a_1, a_2, a_3\}\}$. In fact, a_1, a_2 and a_3 form a cycle of the graph as shown in Fig. 1.

The above example presents that matroids and graphs coincide with each other when the circuit of a matroid is degenerated to the cycle of a graph. The following proposition shows that a matroid can be defined from the viewpoint of circuits.

Proposition 2 [Circuit axiom [14]]. *Let* $\mathbf{C}$ *be a family of subsets of* U. *Then there exists* $M = (U, \mathbf{I})$ *such that* $\mathbf{C} = \mathbf{C}(M)$ *iff* $\mathbf{C}$ *satisfies the following three conditions:*

(C1) $\emptyset \notin \mathbf{C}$;
(C2) If $C_1, C_2 \in \mathbf{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$;
(C3) If $C_1, C_2 \in \mathbf{C}, C_1 \neq C_2$ and $u \in C_1 \bigcap C_2$, *then there exists* $C_3 \in \mathbf{C}$ such that $C_3 \subseteq C_1 \bigcup C_2 - \{u\}$.

The dimension of a vector space and the rank of a matrix are quite useful concepts in linear algebra. The rank function of a matroid is a generalization of these two concepts.

Definition 10 (*Rank function* [14]). Let $M = (U, \mathbf{I})$ be a matroid. The rank function r_M of M is defined as $r_M(X) = max\{|I| : I \subseteq X$, and $I \in \mathbf{I}\}$ for all $X \subseteq U$. $r_M(X)$ is called the rank of X in M.

Based on the rank function of a matroid, one can define the closure operator, which reflects the dependency between a set and elements.

Definition 11 (*Closure* [14]). Let $M = (U, \mathbf{I})$ be a matroid. The closure operator cl_M of M is defined as $cl_M(X) = \{u \in U | r_M(X) = r_M(X \bigcup \{u\})\}$ for all $X \subseteq U$. $cl_M(X)$ is called the closure of X in M.

We say a subset of the ground set is a closed set of a matroid if its closure is equal to itself. In other words, a closed set of a matroid is a fixed point of the closure operator.

Definition 12 (*Closed* set [14]). Let $M = (U, \mathbf{I})$ be a matroid. For all $X \subseteq U, X$ is called a closed set of M if $cl_M(X) = X$.

The following closure axiom shows the connection between matroids and closure operators. In fact, a matroid uniquely determines a closure operator, and vice versa.

Proposition 3 [Closure axiom [14]]. *Let* $cl : 2^U \rightarrow 2^U$ *be an operator. Then there exists a matroid* $M = (U, \mathbf{I})$ *such that* $cl = cl_M$ *iff cl satisfies the following conditions: (CL1) For all* $X \subseteq U, X \subseteq cl(X)$*;(CL2) If* $X \subseteq Y \subseteq U$*, then* $cl(X) \subseteq cl(Y)$*;(CL3) For all* $X \subseteq U, cl(cl(X)) = cl(X)$*; (CL4) For all* $x, y \in U$*, if* $y \in cl(X \bigcup \{x\}) - cl(X)$*, then* $x \in cl(X \bigcup \{y\})$.

This subsection presents the powerful axiomatic system of matroids, which suggests strong compatibility of matroid theory with other theories.

3. An isomorphism from equivalence relations to 2-circuit matroids

In this section, we establish a matroidal structure of rough sets and an isomorphism from equivalence relations to a type of matroids. First of all, we propose approaches to generating a matroid from an equivalence relation as well as to inducing an equivalence relation from a matroid.

3.1. Matroid induced by equivalence relation

This subsection induces a matroid by an equivalence relation, and provides several equivalent formulations of the independent sets of the matroid. Through the circuit axiom of matroids, those sets with only two elements which have a relationship with each other form a matroid.

Definition 13. Let R be an equivalence relation on U. We define a family $\mathbf{C}(R)$ of subsets of U as follows: for all $x, y \in U$ and $x \neq y$,

$$(x, y) \in R \Longleftrightarrow \{x, y\} \in \mathbf{C}(R).$$

Example 4. Let $U = \{a, b, c, d, e\}$ and $R = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (c, d), (d, c), (c, e), (e, c), (d, e), (e, d)\}$. It is straightforward that R is an equivalence relation on U. Then $\mathbf{C}(R) = \{\{a, b\}, \{c, d\}, \{c, e\}, \{d, e\}\}$.

In fact, the family of subsets of the universe induced by an equivalence relation in compliance with the above definition satisfies the circuit axiom. In other words, it determines a matroid.

Proposition 4. *Let* R *be an equivalence relation on* U. *Then* $\mathbf{C}(R)$ *satisfies (C1), (C2) and (C3) of* Proposition 2.

Proof. (C1) and (C2) are straightforward. Let $C_1, C_2 \in C(R), C_1 \neq C_2$ and $x \in C_1 \bigcap C_2$. Without losing generality, let $C_1 = \{x, y\}$ and $C_2 = \{x, z\}$. We know xRy and xRz, which imply that yRz because R is an equivalence relation. Therefore $C_3 = \{y, z\} \in C(R)$, and $C_3 \subseteq C_1 \bigcup C_2 - \{x\}$. This completes the proof. □

According to Proposition 2, there exists a matroid on the universe such that $\mathbf{C}(R)$ is the family of its circuits. Therefore, we establish a matroidal structure for any approximation space.

Definition 14. Let R be an equivalence relation on U. The matroid whose circuit set is $\mathbf{C}(R)$ is denoted by $M(R)=(U,\mathbf{I}(R))$. We say $M(R)=(U,\mathbf{I}(R))$ is the matroid induced by R, where $\mathbf{I}(R)=(Upp(\mathbf{C}(R)))^c$.

The matroid induced by an equivalence relation can be characterized from the viewpoint of rough sets. The following three propositions present three equivalent formulations of the independent sets of the matroid.

Proposition 5. *Let R be an equivalence relation on U and $M(R)=(U,\mathbf{I}(R))$ the matroid induced by R. Then $\mathbf{I}(R)=\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}$.*

Proof. We only need to prove $(Upp(\mathbf{C}(R)))^c=\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}$. For all $X\notin \{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}$, then there exist $x,y\in X$ and $x\neq y$ such that $(x,y)\in R$, i.e., $\{x,y\}\in \mathbf{C}(R)$. Since $\{x,y\}\subseteq X$, $X\in Upp(\mathbf{C}(R))$, which implies $X\notin (Upp(\mathbf{C}(R)))^c$. Hence $(Upp\ (\mathbf{C}(R)))^c\subseteq\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}$. Conversely, for all $X\notin (Upp(\mathbf{C}(R)))^c$, i.e., $X\in Upp(\mathbf{C}(R))$, then there exists $C_X\in \mathbf{C}(R)$ such that $C_X\subseteq X$. Suppose $C_X=\{x,y\}$, then $(x,y)\in R$, which implies $X\notin\{X\subseteq U|\forall x, y\in X, x\neq y\Rightarrow (x,y)\notin R\}$. Hence $\{X\subseteq U|\forall x,y\in X,\ x\neq y\Rightarrow (x,y)\notin R\}\subseteq (Upp(\mathbf{C}(R)))^c$. This completes the proof.

Example 5. (Continued from Example 4). The matroid induced by R is $M(R)=(U,\mathbf{I}(R))$ where $U=\{a,b,c,d,e\}$ and $\mathbf{I}(R)=\{\emptyset,\{a\},\{b\},\{c\},\{d\},\{e\},\{a,c\},\{a,d\},\{a,e\},\{b,c\},\{b,d\},\{b,e\}\}$.

The independent sets of the matroid induced by an equivalence relation can also be characterized by the cardinalities of sets.

Proposition 6. *Let R be an equivalence relation on U and $M(R)=(U,\mathbf{I}(R))$ the matroid induced by R. Then $\mathbf{I}(R)=\{X\subseteq U|\forall x\in U, |X\bigcap RN(x)|\leqslant 1\}$ where $RN(x)=\{y\in U|xRy\}$.*

Proof. According to Proposition 5, we only need to prove $\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\}=\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow\ (x,y)\notin R\}$. For all $X\in\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\}, x,y\in X$ and $x\neq y$, then $(x,y)\notin R$. In fact, if there exist $x,y\in X, x\neq y$ such that $(x,y)\in R$, then $\{x,y\}\subseteq RN(x)=\{y\in U|xRy\}$. Thus $|X\bigcap RN(x)|\geqslant 2$, which is contradictory with $|X\bigcap RN(x)|\leqslant 1$. Hence $X\in\{X\subseteq U|\forall x, y\in X, x\neq y\Rightarrow (x,y)\notin R\}$; in other words, $\{X\subseteq U|\forall x\in U, |X\bigcap RN(x)|\leqslant 1\}\subseteq\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}$. Conversely, for all $X\in\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}, y\notin RN(x)$ for all $y\in X$ and $x\neq y$, then $|X\bigcap RN(x)|\leqslant 1$ for all $x\in X$. We now prove that $|X\bigcap RN(x)|\leqslant 1$ for all $x\in X^c$. If there exists $y\in X^c$ such that $|X\bigcap RN(y)|\geqslant 2$, then we suppose $\{z,h\}\subseteq X\bigcap RN(y)$, hence $(y,z)\in R$ and $(y,h)\in R$, which imply $(z,h)\in R$. Since $\{z,h\}\subseteq X$, it is contradictory that $(x,y)\notin R$ for all $x,y\in X$ and $x\neq y$. Therefore, $\{X\subseteq U|\forall x,y\in X, x\neq y\Rightarrow (x,y)\notin R\}\subseteq\{X\subseteq U|\forall x\in U,\ |X\bigcap RN(x)|\leqslant 1\}$.

In the following proposition, we connect the approximations with the independent sets of the matroid induced by an equivalence relation. For this purpose, two lemmas are presented.

Lemma 1. *Let R be an equivalence relation on U. Then $Max\{X\subseteq U|\forall x\in U, |X\bigcap RN(x)|\leqslant 1\}=\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}$.*

Lemma 2. *Let R be an equivalence relation on U. Then $Low(Max\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\})=\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\}$.*

An equivalent formulation of the independent sets of the matroid induced by an equivalence relation is provided from the viewpoint of the upper approximation. In fact, a subset of a universe is an independent set if and only if it is contained in another set whose upper approximation is equal to the universe, and which is minimal.

Proposition 7. *Let R be an equivalence relation on U and $M(R)=(U,\mathbf{I}(R))$ the matroid induced by R. Then $\mathbf{I}(R)=Low(Min\{X\subseteq U|\overline{R}(X)=U\})$.*

Proof. According to Proposition 6 and Lemma 1, we only need to prove $Min\{X\subseteq U|\overline{R}(X)=U\}=\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}$, because $Low(Min\{X\ \ \subseteq U|\overline{R}(X)=U\})=Low(\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\})=Low(Max\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\})=\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|\leqslant 1\}=\mathbf{I}(R)$. In fact, for all $X\in\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}$, since for all $x\in U,|X\bigcap RN(x)|=1$ which implies $X\bigcap RN(x)\neq\emptyset,\overline{R}(X)=U$. For all $x\in X$, since $|X\bigcap RN(x)|=1,(X-\{x\})\bigcap RN(x)=\emptyset$ and $\overline{R}(X-\{x\})\subseteq U-\{x\}$ which implies $\overline{R}(X-\{x\})\neq U$. Hence $X\in Min\{X\subseteq U|\overline{R}(X)=U\}$. Therefore, $\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}\subseteq Min\{X\subseteq U|\overline{R}(X)=U\}$. Conversely, for all $X\in Min\{X\subseteq U|\overline{R}(X)=U\},\overline{R}(X)=U$, then for all $x\in U,X\bigcap RN(x)\neq\emptyset$, which implies $|X\bigcap RN(x)|\geqslant 1$. If there exists $y\in U$ such that $|X\bigcap RN(y)|>1$, i.e., $|X\bigcap RN(y)|\geqslant 2$, then we suppose $\{g,h\}\subseteq X\bigcap RN(y)$, so $\overline{R}(X-\{g\})=U$, which is contradictory with $X\in Min\{X\subseteq U|\overline{R}(X)=U\}$. Hence $X\in\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}$. Therefore, $Min\{X\subseteq U|\overline{R}(X)=U\}\subseteq\{X\subseteq U|\forall x\in U,|X\bigcap RN(x)|=1\}$. This completes the proof. □

The above proposition shows an explanation of the upper approximation using independent sets of a matroid. Similarly, owing to the duality, the independent sets of the matroid induced by an equivalence relation can also be represented by the lower approximations.

Corollary 1. *Let R be an equivalence relation on U and $M(R)=(U,\mathbf{I}(R))$ the matroid induced by R. Then $\mathbf{I}(R)=Low(Min\{X\subseteq U|\underline{R}(X^c)=\emptyset\})$.*

Proof. According to Proposition 7, $\mathbf{I}(R)=Low(Min\{X\subseteq U|\overline{R}(X)=U\})=\quad Low(Min\{X\subseteq U|[\underline{R}(X^c)]^c=U\})=Low(Min\{X\subseteq U|\underline{R}(X^c)=\emptyset\})$. □

These equivalent formulations of the independent sets of the matroid induced by an equivalence relation lay sound foundations for investigating rough sets from a matroidal point of view.

3.2. Equivalence relation induced by matroid

In this subsection, we consider a converse question, that is, how a matroid induces an equivalence relation, to establish a deep correspondence between equivalence relations and matroids.

Definition 15. Let $M=(U,\mathbf{I})$ be a matroid. One can define a relation $R(M)$ on U as follows: for all $x,y\in U$,

$$(x,y)\in R(M)\Longleftrightarrow x=y or \exists C\in\mathbf{C}(M) such that \{x,y\}\subseteq C.$$

In fact, according to the above definition, the relation induced by a matroid is an equivalence relation.

Proposition 8. *Let $M=(U,\mathbf{I})$ be a matroid. Then $R(M)$ is an equivalence relation on U.*

Definition 15 and Proposition 8 show an induction of equivalence relation by a matroid. SubSection 3.1 proposes an approach to inducing a matroid from an equivalence relation. The following proposition builds the connection between these two inductions.

Proposition 9. *Let R be an equivalence relation on U. Then* $R(M(R)) = R$.

The above proposition indicates that these two inductions are converse. In fact, a close relationship between equivalence relations and matroids can be further reflected. For this reason, a type of matroid called 2-circuit matroid is defined.

Definition 16 (*2-circuit matroid*). Let $M = (U, \mathbf{I})$ be a matroid. If for all $C \in \mathbf{C}(M), |C| = 2$, then we say M is a 2-circuit matroid.

In fact, the matroid induced by an equivalence relation is a 2-circuit matroid.

Proposition 10. *Let R be an equivalence relation on U. Then* $M(R) = (U, \mathbf{I}(R))$ *is a 2-circuit matroid.*

The following proposition also shows that these two inductions are converse. According to Proposition 8, any matroid induces an equivalence relation, and the matroid induced by the equivalence relation is equal to the original matroid.

Proposition 11. *Let* $M = (U, \mathbf{I})$ *be a 2-circuit matroid. Then* $M(R(M)) = M$.

The above results establish a one-to-one correspondence between equivalence relations and 2-circuit matroids. Furthermore, we construct an isomorphism from equivalence relations to 2-circuit matroids. For this purpose, the isomorphism is introduced.

Definition 17 (*Isomorphism* [35]). Let $(\mathbb{A}, \diamondsuit)$ and $(\mathbb{B}, \circ)$ be two closed algebraic systems. If there exists a bijection h from $\mathbb{A}$ to $\mathbb{B}$ such that

$$h(A_1 \diamondsuit A_2) = h(A_1) \circ h(A_2) \text{ for all } A_1, A_2 \in \mathbb{A},$$

then we say h is an isomorphism, and $\mathbb{A}, \mathbb{B}$ are isomorphic, denoted by $\mathbb{A} \cong \mathbb{B}$.

We consider two closed algebraic systems $(\mathbb{R}, \bigcap)$ where $\mathbb{R}$ is the family of all equivalence relations on U, and $(\mathbb{C}, \bigcap)$ where $\mathbb{C}$ is the family of circuit sets of all 2-circuit matroids. A mapping from one algebra system to another is constructed.

Definition 18. An operator $h : \mathbb{R} \to \mathbb{C}$ is defined as follows: for all $R \in \mathbb{R}$,

$$h(R) = \mathbf{C}(R).$$

The mapping constructed in the above definition is an isomorphism; in other words, these two algebra systems are isomorphic.

Theorem 1. *h is an isomorphism from* $\mathbb{R}$ *to* $\mathbb{C}$*, i.e.,* $\mathbb{R} \cong \mathbb{C}$.

Proof. First, we prove h is a bijection, that is, h is injective and surjective. It is straightforward that h is injective. For all $\mathbf{C} \in \mathbb{C}$, we suppose $R = R(M)$, where M expresses the matroid whose circuit set is $\mathbf{C}$. According to Proposition 11, $M(R(M)) = M$; in other words, the circuit set of $M(R(M))$ is $\mathbf{C}$. Thus h is surjective. Second, we prove $h(R_1 \bigcap R_2) = h(R_1) \bigcap h(R_2)$, i.e., $\mathbf{C}(R_1 \bigcap R_2) = \mathbf{C}(R_1) \bigcap \mathbf{C}(R_2)$ for all $R_1, R_2 \in \mathbb{R}$. For all $\{x, y\} \in \mathbf{C}(R_1 \bigcap R_2)$, then $(x, y) \in R_1 \bigcap R_2$. Thus $(x, y) \in R_1$ and $(x, y) \in R_2$, i.e., $\{x, y\} \in \mathbf{C}(R_1)$ and $\{x, y\} \in \mathbf{C}(R_2)$. Hence $\mathbf{C}(R_1 \bigcap R_2) \subseteq \mathbf{C}(R_1) \bigcap \mathbf{C}(R_2)$. Conversely, for all $\{x, y\} \in \mathbf{C}(R_1) \bigcap \mathbf{C}(R_2), \{x, y\} \in \mathbf{C}(R_1)$ and $\{x, y\} \in \mathbf{C}(R_2)$, i.e., $(x, y) \in R_1$ and $(x, y) \in R_2$, then $(x, y) \in R_1 \bigcap R_2$, i.e., $\{x, y\} \in \mathbf{C}(R_1 \bigcap R_2)$. Hence $\mathbf{C}(R_1) \bigcap \mathbf{C}(R_2) \subseteq \mathbf{C}(R_1 \bigcap R_2)$. This completes the proof.

In general, isomorphic mathematical structures can be regarded as the similarity. Therefore, through the isomorphism from equivalence relations to 2-circuit matroids, the study for rough sets is converted to the study for 2-circuit matroids.

4. Matroidal approaches to rough sets

This section provides equivalent formulations of some important concepts in rough sets with matroidal approaches. Specifically, the upper approximation operator is characterized by the closure operator of the matroid induced by an equivalence relation. For this purpose, a lemma is presented.

Lemma 3 *[14]. Let* $M = (U, \mathbf{I})$ *be a matroid. Then for all* $X \subseteq U$,

$$cl_M(X) = X \bigcup \{u \in U | \exists C \in \mathbf{C}(M), \text{such that } u \in C \subseteq X \bigcup \{u\}\}.$$

The following proposition establishes a relationship between the upper approximation operator with respect to an equivalence relation and the closure operator of the corresponding matroid.

Proposition 12. *Let R be an equivalence relation on U. Then* $\overline{R}(X) = cl_{M(R)}(X)$ *for all* $X \subseteq U$.

Proof. According to Definition 14 and Lemma 3, $cl_{M(R)}(X) = X \bigcup \{x \in U | \exists C \in \mathbf{C}(R)$, such that $x \in C \subseteq X \bigcup \{x\}\} = X \bigcup \{x \in U | \exists y \in X, xRy$, such that $\{x, y\} \subseteq X \bigcup \{x\}\} = \{x \in U | \exists y \in X, xRy\} = \{x \in U | RN(x) \bigcap X \neq \emptyset\} = \overline{R}(X)$. This completes the proof.

The above proposition shows that the upper approximation operator is equal to the closure operator of the matroid induced by an equivalence relation. Similarly, the lower approximation operator can also be represented by the closure operator.

Corollary 2. *Let R be an equivalence relation on U. Then* $\underline{R}(X) = [cl_{M(R)}(X^c)]^c$ *for all* $X \subseteq U$.

Corollary 3. *Let R be an equivalence relation on U. Then for all* $x \in U, \overline{R}(\{x\}) = RN(x) = cl_{M(R)}(\{x\})$.

Furthermore, some other concepts related to the approximation operators are also described by the closure operator. For example, the precise set in rough sets is characterized by the closed set of the matroid.

Corollary 4. *Let R be an equivalence relation on U. For all* $X \subseteq U, X$ *is a R-precise set iff X is a closed set of* $M(R)$.

Similarly, the rough set can be represented by the closed set of the matroid. In fact, a subset of a universe is a rough set if and only if it is not a closed set of the matroid.

Corollary 5. *Let R be an equivalence relation on U. For all* $X \subseteq U, X$ *is a R-rough set iff X is not a closed set of* $M(R)$.

We can also use the closure operator to represent a relation when it is an equivalence relation. In fact, a relation is an equivalence one if and only if the upper approximation operator satisfies the closure axioms.

Proposition 13. *Let R be a relation on U. Then R is an equivalence relation iff* $\overline{R}$ *satisfies (CL1), (CL2), (CL3) and (CL4) of Proposition 3, where* $\overline{R}(X) = \{x \in U | RN(x) \bigcap X \neq \emptyset\}$ *for all* $X \subseteq U$ *and* $RN(x) = \{y \in U | xRy\}$.

Proof. ($\Rightarrow$): According to Proposition 12, it is straightforward. ($\Leftarrow$): Since $\overline{R}$ satisfies (CL1) and (CL3), according to literature [41], R is reflexive and transitive. Since $\overline{R}$ satisfies (CL4), for all $x, y \in U$, $y \in \overline{R}(X \bigcup \{x\}) - \overline{R}(X), y \in \overline{R}(X \bigcup \{y\})$. For all $(x, y) \in R, y \in RN(x)$, i.e., $x \in \overline{R}(\{y\}) = \overline{R}(\{y\} \bigcup \emptyset) - \overline{R}(\emptyset)$, then $y \in \overline{R}(\{x\} \bigcup \emptyset) = \overline{R}(\{x\})$, i.e., $(y, x) \in R$. Therefore, R is symmetric. To sum up, this completes the proof. □

The above results show that many important concepts in rough sets can be concisely characterized by corresponding concepts in matroid theory. Therefore, matroids may be efficient to study rough sets.

5. Matroidal approaches to attribute reduction

Attribute reduction, also called feature selection, is to preserve the essence and remove the redundancy of an information/decision table [3,8]. In this section, we provide two equivalent characterizations of attribute reduction of information systems in the matroidal structure. First of all, the quotient set, this is, the family of all equivalence classes of an equivalence relation is represented by the rank function of the matroid induced by the equivalence relation. In fact, an equivalence class is a maximal set whose rank is equal to one.

Proposition 14. *Let R be an equivalence relation on U. Then*

$$U/R = Max\{X \subseteq U | r_{M(R)}(X) = 1\},$$

where U/R is the family of all equivalence classes of R, i.e., $U/R = \{RN(x) | x \in U\}$.

Proof. We only need to prove $\{RN(x)|x \in U\} = Max\{X \subseteq U | r_{M(R)}(X) = 1\}$. For all $x \in U$, it is straightforward that $r_{M(R)}(RN(x)) = 1$. If $RN(x) \notin Max\ \{X \subseteq U | r_{M(R)}(X) = 1\}$, then there exists $Y \subseteq U$ such that $RN(x) \subset Y$ and $Y \in Max\{X \subseteq U | r_{M(R)}(X) = 1\}$. We suppose $y \in Y - RN(x)$, then $RN(x) \bigcup \{y\} \subseteq Y$. According to Proposition 6, $r_{M(R)}(RN(x) \bigcup \{y\}) = 2$, then $r_{M(R)}(Y) \geqslant r_{M(R)}(R)(RN(x) \bigcup \{y\}) = 2$, which is contradictory with $r_{M(R)}(Y) = 1$. Hence $RN(x) \in Max\ \{X \subseteq U | r_{M(R)}(X) = 1\}$, i.e., $U/R \subseteq Max\{X \subseteq U | r_{M(R)}(X) = 1\}$. Conversely, for all $X \in Max\ \{X \subseteq U | r_{M(R)}(X) = 1\}$, we suppose $x \in X$, then we now prove $X = RN(x)$. In fact, if $X \neq RN(x)$, then $X - RN(x) \neq \emptyset$ or $RN(x) - X \neq \emptyset$. If $X - RN(x) \neq \emptyset$, then we suppose $y \in X - RN(x)$, thus $\{x, y\} \subseteq X$ and $r_{M(R)}(\{x, y\}) = 2$, which is contradictory with $r_{M(R)}(X) = 1$. If $RN(x) - X \neq \emptyset$, then we suppose $y \in RN(x) - X$, thus $X \subseteq X \bigcup \{y\}$ and $r_{M(R)}(X \bigcup \{y\}) = 1$, which is contradictory with $X \in Max\ \{X \subseteq U | r_{M(R)}(X) = 1\}$. Therefore, $Max\{X \subseteq U | r_{M(R)}(X) = 1\} \subseteq U/R$. □

The indiscernibility relation in an information system is also an equivalence relation, and an equivalence relation and its quotient set are uniquely determined by each other. Therefore, Proposition 14 lays a sound foundation for studying attribute reduction of information systems in the matroidal structure. In fact, the quotient set of an equivalence relation can also be equivalently formulated by the closure operator of the matroid.

Proposition 15. *Let R be an equivalence relation on U. Then*

$$U/R = \{cl_{M(R)}(X) | X \subseteq U, r_{M(R)}(X) = 1\} = \{cl_{M(R)}(\{x\}) | x \in U\}.$$

Proof. According to Corollary 3, it is straightforward. □

The next proposition is presented to establish a relationship between the equivalence relation generated by the intersection of two equivalence relations and the matroids induced by these two equivalence relations respectively.

Proposition 16. *Let R_1 and R_2 be two equivalence relations on U. Then*

$$U/(R_1 \bigcap R_2) = Max\{X \subseteq U | r_{M(R_1)}(X) = r_{M(R_2)}(X) = 1\}.$$

In fact, the relationship between the equivalence relation induced by the intersection of two equivalence relations and the matroids induced by these equivalence relations respectively can also be reflected by the closure operator.

Proposition 17. *Let R_1 and R_2 be two equivalence relations on U. Then*

$$U/(R_1 \bigcap R_2) = \{cl_{M(R_1)}(\{x\}) \bigcap cl_{M(R_2)}(\{x\}) | x \in U\}.$$

The following proposition provides a sufficient and necessary condition for the rank function of the matroid induced by an equivalence relation.

Proposition 18. *Let R_1 and R_2 be two equivalence relations on U. Then $Max\{X \subseteq U | r_{M(R_1)}(X) = 1\} = Max\{X \subseteq U | r_{M(R_2)}(X) = 1\}$ iff $\{X \subseteq U | r_{M(R_1)}(X) = 1\} = \{X \subseteq U | r_{M(R_2)}(X) = 1\}$.*

Proof. It is straightforward. In fact, $\{X \subseteq U | r_{M(R)}(X) = 1\} = Low(Max\{X \subseteq U | r_{M(R)}(X) = 1\}) - \{\emptyset\}$ for any equivalence relation R. □

The following proposition indicates that the indiscernibility relation induced by any attribute subset can be represented by the rank function of the matroid induced by each attribute subset.

Proposition 19. *Let $IS = (U, A)$ be an information system. Then for all $B \subseteq A$,*

$$IND(B) = Max\{X \subseteq U | \forall b \in B, r_{M(b)}(X) = 1\}.$$

Proof. According to Proposition 16, it is straightforward. □

Note that b is used to express $IND(\{b\})$; in other words, b denotes the equivalence relation induced by attribute subset $\{b\}$. Based on Proposition 19, we present equivalent formulations of attribute reduction of information systems.

Theorem 2. *Let $IS = (U, A)$ be an information system. For all $B \subseteq A$, B is a reduct of IS iff it satisfies the following two conditions:*

(1) $\forall b \in B, \{X \subseteq U | \forall c \in B - \{b\}, r_{M(c)}(X) = 1\} \neq \{X \subseteq U | \forall c \in B, r_{M(c)}(X) = 1\}$;
(2) $\{X \subseteq U | \forall b \in B, r_{M(b)}(X) = 1\} = \{X \subseteq U | \forall a \in A, r_{M(a)}(X) = 1\}$.

Theorem 2 shows that attribute reduction of an information system keeps those sets with one rank unchanged from a matroidal point of view. That provides a new view for attribute reduction in information systems. In fact, from the viewpoint of closure operators, attribute reduction of information systems can also be represented.

Lemma 4. *Let $IS = (U, A)$ be an information system. For all $B \subseteq A$, B is a reduct of IS iff it satisfies the following two conditions:*

(1) $\forall b \in B, \{\bigcap_{c \in B} cl_{M(c)}(\{x\}) | x \in U\} \neq \{\bigcap_{c \in B-\{b\}} cl_{M(c)}(\{x\}) | x \in U\}$;
(2) $\{\bigcap_{c \in B} cl_{M(c)}(\{x\}) | x \in U\} = \{\bigcap_{c \in A} cl_{M(c)}(\{x\}) | x \in U\}$.

In fact, closure operators of the matroids induced by equivalence relations generated by some attributes can also be used to equivalently characterize attribute reduction.

Theorem 3. *Let $IS = (U, A)$ be an information system. For all $B \subseteq A$, B is a reduct of IS iff it satisfies the following two conditions:*

(1) $\forall b \in B$, there exists $x_b \in U$, such that

$$\bigcap_{c \in B} cl_{M(c)}(\{x_b\}) \neq \bigcap_{c \in B-\{b\}} cl_{M(c)}(\{x_b\});$$

(2) $\forall x \in U, \bigcap_{c \in B} cl_{M(c)}(\{x\}) = \bigcap_{c \in A} cl_{M(c)}(\{x\})$.

This section points out two equivalent formulations of attribute reduction from viewpoints of closure operator and rank function respectively. Matroids have a powerful axiomatic system, therefore more equivalent characterizations for attribute reduction may be available. These matroidal representations for attribute reduction present new perspectives to study information systems.

6. Conclusions

This paper establishes a matroidal structure of rough sets and employs it to equivalently characterize rough sets and information systems. On one hand, we construct an isomorphism from equivalence relations to 2-circuit matroids, which lays sound foundations for integrating rough sets with matroids. On the other hand, matroidal approaches are used to study rough sets and information systems. In rough sets, important concepts including approximation operators and precise sets are equivalently represented by matroids. In information systems, attribute reduction is also characterized in the matroidal structure.

Though some works have been conducted in this paper, there are still many interesting topics deserving further investigation:

(1) Rough set characteristic discovering. Equivalent characterizations of concepts in rough sets using corresponding matroidal concepts could be much helpful to reveal the intrinsic characteristics of rough sets. For example, an isomorphism from equivalence relations to 2-circuit matroids provides matroidal platforms to study rough set problems.
(2) Attribute reduction algorithm designing. Some quantitative tools such as attribute significance, information entropy and approximate classified precision serve as heuristic functions of heuristic algorithms for attribute reduction. Similarly, some quantitative tools in matroid theory including rank function and matroid connectivity could work better because of sound theoretical foundations.
(3) Cost-sensitive rough set investigating. Matroids construct good platforms for greedy algorithms, especially some problems containing weight functions. Matroids could provide well-established mathematical structures to characterize and solve problems in cost-sensitive rough sets [20].

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